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THE VARIETY GENERATED BY AN AI-SEMIRING OF ORDER THREE

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Abstract: Up to isomorphism, there are 61 ai-semirings of order three. The finite basis problem for these semirings is investigated. This problem for 45 semirings of them is answered by some results in the literature. The remaining semirings are studied using equational logic. It is shown that with the possible exception of the semiring S_7, all ai-semirings of order three are finitely based.

Keywords: Ai-semiring, Identity, Finitely based variety.

Introduction and preliminaries

By a *variety* we mean a class of algebras of the same type that is closed under subalgebras, homomorphic images and direct products. It is well-known (Birkhoff's theorem) that a class of algebras of the same type is a variety if and only if it is an equational class. One of the fundamental problems about a variety is the so called finite basis problem, that is, whether it can be defined by finitely many identities. If the answer is positive, then it is called *finitely based*. Otherwise, it is called *nonfinitely based*. An algebra A is said to be finitely based (resp., nonfinitely based) if the variety generated by A is finitely based (resp., nonfinitely based).

In 1951 Lyndon [9] showed that all two-element algebras are finitely based and formulated the problem whether every finite algebra is finitely based. This problem has been answered negatively, since a certain seven-element groupoid [10] was shown to be nonfinitely based. Some classical algebras are finite based. For example, so are every finite group [15], every finite associative ring [6, 8], every finite lattice [11] and every commutative semigroup [18]. However, not every finite semigroup and not every finite semiring are finitely based. The first example of an nonfinitely based finite semigroup (resp., semiring) has been given by Perkins [18] (resp., Dolinka [1]).

To seek a ultimate solution to the finite basis problem for finite algebras, Tarski [24] proposed the following problem: Is there an algorithm to decide whether a finite algebra is finitely based? McKenzie [12] negatively answered this problem for finite groupoids. However, this problem is still open when restricted to finite semigroups and finite semirings.

By a *semiring* we mean an algebra $(S, +, \cdot)$ such that

- the additive reduct $(S, +)$ is a commutative semigroup;
- the multiplicative reduct $(S, \cdot)$ is a semigroup;
- $(S, +, \cdot)$ satisfies the identities $x(y + z) \approx xy + xz$ and $(y + z)x \approx yx + zx$.

One can easily find many examples of semirings in almost all branches of mathematics. Semirings can be regarded as a common generalization of both rings and distributive lattices. They have been widely applicated in theoretical computer science and information science. We shall say that a semiring is an *additively idempotent semiring* (ai-semiring for short) if its additive reduct is a semilattice, i.e., a commutative idempotent semigroup. The variety of all ai-semirings is denoted by **AI**. Let $P_f(X^+)$ denote the set of all finite non-empty subsets of the free semigroup X^+ on a countably infinite set X of variables. If we define an addition and a multiplication on $P_f(X^+)$ by

$$A + B = A \cup B, \quad A \circ B = \{ab \mid a \in A, b \in B\},$$

then $(P_f(X^+), +, \circ)$ is free in **AI** with respect to the mapping $\varphi\colon X \to P_f(X^+), x \mapsto \{x\}$ (see [7, Theorem 2.5]). An ai-semiring identity (**AI**-identity for short) over X is an expression of the form $u \approx v$, where $u, v \in P_f(X^+)$. For convenience, we write $u_1 + u_2 + \cdots + u_k \approx v_1 + v_2 + \cdots + v_\ell$ for the ai-semiring identity $\{u_i \mid 1 \leq i \leq k\} \approx \{v_j \mid 1 \leq j \leq \ell\}$.

In the last decades, several authors studied the finite basis problem for various semiring varieties. There is a rich literature on this subject (see [1–5, 16, 17, 19–23, 25, 27, 28]). Dolinka [1] found the first example of a finite nonfinitely based ai-semiring. In [2] he provided a sufficient condition under which an ai-semiring is inherently nonfinitely based, i.e., the variety **V** generated by this semiring is locally finite and every locally finite variety **W** for which $\mathbf{V} \subseteq \mathbf{W}$ is nonfinitely based. As an application, it was shown in [3, 4] that some ai-semirings are inherently nonfinitely based[1]. McKenzie and Romanowska [13] showed that all ai-semirings satisfying $x^2 \approx x$ and $xy \approx yx$ are finitely based. Zhao et al. [27, 28] considered the finite basis problem for ai-semirings satisfying $x^2 \approx x$ that are related to Green's relations. Based on the work of [13, 27, 28], Ghosh et al. [5] and Pastijn [16] proved that all ai-semirings satisfying $x^2 \approx x$ are finitely based. Ren et al. [21] showed that this result holds for all ai-semirings satisfying $x^3 \approx x$. However, not every ai-semirings satisfying $x^n \approx x$ $(n \geq 4)$ is finitely based (see [22]). Recently, Ren et al. [20] answered the finite basis problem for ai-semirings satisfying $x^n \approx x$ and $xy \approx yx$ in which $n-1$ is square-free. From these references one can find that semirings of small order have played an important role. This motivates some authors to investigate the finite basis problem for ai-semirings of small order. In this direction, Shao and Ren [23] considered the variety generated by all ai-semirings of order two. Vechtomov and Petrov [25] studied the variety generated by all semirings of order two whose multiplicative reduct is a semilattice. Moreover, McNulty and Willard [14] initiated the study of the finite basis problem for algebras of order three. The present paper follows this line of investigation. We shall systematically study the finite basis problem for ai-semirings of order three. For this, the following information about ai-semirings of order two in [23] are necessary.

Up to isomorphism, there are exactly 6 ai-semirings of order two, which are listed as L_2, R_2, M_2, D_2, N_2 and T_2 in Table 1. We assume that the underlying set of each of these semirings is $\{0, 1\}$. Their Cayley tables for addition and multiplication are listed in the 2nd and respectively the 3rd columns of Table 1 while the 4th column contains their equational bases.

To present the solution of the word problem for ai-semirings of order two, we need to introduce the following notations. Let ω be an element of X^+ and x an element of ω. Then

- ◇ $c(\omega)$ denotes the *content* of ω, i.e., the set of all variables occurring in ω.
- ◇ $h(\omega)$ denotes the *head* of ω, i.e., the first variable occurring in u.
- ◇ $t(\omega)$ denotes the *tail* of ω, i.e., the last variable occurring in u.
- ◇ $\ell(\omega)$ denotes the *length* of ω, i.e., is the number of variables occurring in u, where each letter is counted as many times as it occurs in u.
- ◇ $m(x,\ \omega)$ denotes the *multiplicity* of x in ω, i.e., the number of occurrences of x in w.

[1]The semiring varieties in Dolinka's papers are types of $(2, 2, 0)$.

Table 1. The 2-element ai-semirings

Semiring	Addition	Multiplication	Equational basis
L_2	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}0&0\\1&1\end{matrix}$	$xy \approx x$
R_2	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}0&1\\0&1\end{matrix}$	$xy \approx y$
M_2	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}0&1\\1&1\end{matrix}$	$x + y \approx xy$
D_2	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}0&0\\0&1\end{matrix}$	$x^2 \approx x,\ xy \approx yx,\ x + xy \approx x$
N_2	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}0&0\\0&0\end{matrix}$	$xy \approx zt,\ x + x^2 \approx x$
T_2	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}1&1\\1&1\end{matrix}$	$xy \approx zt,\ x + x^2 \approx x^2$

The following result follows from [23, Lemma 1.1]. We shall directly apply it without further notice.

Lemma 1. *Let $u \approx v$ be an nontrivial* **AI***-identity, where $u = u_1 + \cdots + u_k, v = v_1 + \cdots + v_\ell, u_i, v_j \in X^+, 1 \le i \le k, 1 \le j \le \ell$. Then*

(i) *L_2 satisfies $u \approx v$ if and only if $\{h(u_i) \mid 1 \le i \le k\} = \{h(v_j) \mid 1 \le j \le \ell\}$;*

(ii) *R_2 satisfies $u \approx v$ if and only if $\{t(u_i) \mid 1 \le i \le k\} = \{t(v_j) \mid 1 \le j \le \ell\}$;*

(iii) *M_2 satisfies $u \approx v$ if and only if $\bigcup\{c(u_i) \mid 1 \le i \le k\} = \bigcup\{c(v_j) \mid 1 \le j \le \ell\}$;*

(iv) *D_2 satisfies $u \approx v$ if and only if $(\forall u_i \in u)(\exists v_j \in v)c(v_j) \subseteq c(u_i)$ and $(\forall v_k \in v)(\exists u_\ell \in u)c(u_\ell) \subseteq c(v_k)$;*

(v) *N_2 satisfies $u \approx v$ if and only if $\{u_i \in u \mid \ell(u_i) = 1\} = \{v_j \in v \mid \ell(v_j) = 1\}$;*

(vi) *T_2 satisfies $u \approx v$ if and only if $\{u_i \in u \mid \ell(u_i) \ge 2\} \ne \emptyset, \{v_j \in v \mid \ell(v_j) \ge 2\} \ne \emptyset$.*

Up to isomorphism, there are 61 ai-semirings of order three[2], which are listed as S_i, $1 \le i \le 61$ in Table 2. We assume that the carrier set of each of these semirings is $\{1, 2, 3\}$. Their Cayley tables for addition and multiplication are listed in Table 2. It is easy to check that there are 24 ai-semirings of order three satisfying $x^3 \approx x$. By the main results of [21] we have that these semirings are all finitely based. So we only need to study the finite basis problem for the remaining 37 semirings. In fact, some of these semirings are members of the variety which are generated by all ai-semirings of order two. By the the main result of [23] it follows that they are all finitely based. Thus we have

Proposition 1. *The following ai-semirings are finitely based:* S_1, S_3, S_5, S_8, S_9, S_{10}, S_{11}, S_{12}, S_{13}, S_{14}, S_{15}, S_{16}, S_{17}, S_{18}, S_{19}, S_{20}, S_{21}, S_{22}, S_{23}, S_{24}, S_{25}, S_{26}, S_{27}, S_{28}, S_{29}, S_{30}, S_{31}, S_{32}, S_{33}, S_{34}, S_{35}, S_{36}, S_{37}, S_{38}, S_{39}, S_{40}, S_{41}, S_{42}, S_{43}, S_{48}, S_{49}, S_{50}, S_{51}, S_{52} *and* S_{61}.

For an ai-semiring S, S^* denotes the (multiplicative) left-right dual of S. It is easy to see that if S is finitely based, so is S^*. Thus, in the remaining we only need to study the finite basis problem for S_2, S_4, S_7, S_{44}, S_{46}, S_{47}, S_{53}, S_{55}, S_{57}, S_{58}, S_{59} and S_{60}. The following theorem is our main result.

Theorem 1. *With the possible exception of S_7, all ai-semirings of order three are finitely based.*

[2] We wrote a program and obtained this result.

Table 2. The 3-element ai-semirings

Semiring	+			·			Semiring	+			·		
	1	1	1	1	1	1		1	1	1	1	1	1
S_1	1	2	1	1	1	1	S_2	1	2	1	1	1	1
	1	1	3	1	1	1		1	1	3	1	1	2
	1	1	1	1	1	1		1	1	1	1	1	1
S_3	1	2	1	1	1	1	S_4	1	2	1	1	1	1
	1	1	3	1	1	3		1	1	3	1	2	3
	1	1	1	1	1	1		1	1	1	1	1	1
S_5	1	2	1	1	1	1	S_6	1	2	1	1	1	2
	1	1	3	3	3	3		1	1	3	1	1	3
	1	1	1	1	1	1		1	1	1	1	1	1
S_7	1	2	1	1	1	2	S_8	1	2	1	1	2	1
	1	1	3	1	2	3		1	1	3	1	1	3
	1	1	1	1	1	1		1	1	1	1	1	1
S_9	1	2	1	1	2	1	S_{10}	1	2	1	1	2	3
	1	1	3	3	3	3		1	1	3	1	3	2
	1	1	1	1	1	1		1	1	1	1	1	3
S_{11}	1	2	1	2	2	2	S_{12}	1	2	1	1	1	3
	1	1	3	3	3	3		1	1	3	1	1	3
	1	1	1	1	1	3		1	1	1	1	1	3
S_{13}	1	2	1	1	1	3	S_{14}	1	2	1	1	2	3
	1	1	3	3	3	3		1	1	3	1	1	3
	1	1	1	1	1	1		1	1	1	1	2	3
S_{15}	1	2	1	1	2	1	S_{16}	1	2	1	1	2	3
	1	1	3	3	3	3		1	1	3	1	2	3
	1	1	1	2	2	2		1	1	3	1	1	1
S_{17}	1	2	1	2	2	2	S_{18}	1	2	3	1	1	1
	1	1	3	2	2	2		3	3	3	1	1	1
	1	1	3	1	1	1		1	1	3	1	1	1
S_{19}	1	2	3	1	1	1	S_{20}	1	2	3	1	1	1
	3	3	3	1	1	3		3	3	3	3	3	3
	1	1	3	1	1	1		1	1	3	1	1	1
S_{21}	1	2	3	1	2	1	S_{22}	1	2	3	1	2	1
	3	3	3	1	1	1		3	3	3	1	1	3
	1	1	3	1	1	1		1	1	3	1	1	1
S_{23}	1	2	3	1	2	1	S_{24}	1	2	3	2	2	2
	3	3	3	3	3	3		3	3	3	1	1	1
	1	1	3	1	1	1		1	1	3	1	1	1
S_{25}	1	2	3	2	2	2	S_{26}	1	2	3	2	2	2
	3	3	3	1	1	3		3	3	3	3	3	3
	1	1	3	1	1	3		1	1	3	1	1	3
S_{27}	1	2	3	1	1	3	S_{28}	1	2	3	1	1	3
	3	3	3	1	1	3		3	3	3	3	3	3
	1	1	3	1	1	3		1	1	3	1	1	3
S_{29}	1	2	3	1	2	3	S_{30}	1	2	3	1	2	3
	3	3	3	1	1	3		3	3	3	3	3	3

	1	1	3	1	1	3		1	1	3	1	2	1
S_{31}	1	2	3	2	2	3	S_{32}	1	2	3	1	2	1
	3	3	3	3	3	3		3	3	3	1	2	1
	1	1	3	1	2	1		1	1	3	1	2	1
S_{33}	1	2	3	1	2	1	S_{34}	1	2	3	2	2	2
	3	3	3	1	2	3		3	3	3	1	2	1
	1	1	3	1	2	1		1	1	3	1	2	1
S_{35}	1	2	3	2	2	2	S_{36}	1	2	3	2	2	2
	3	3	3	1	2	3		3	3	3	3	2	3
	1	1	3	1	2	3		1	1	3	1	2	3
S_{37}	1	2	3	1	2	3	S_{38}	1	2	3	1	2	3
	3	3	3	1	2	3		3	3	3	3	3	3
	1	1	3	1	2	3		1	1	3	1	2	3
S_{39}	1	2	3	2	2	2	S_{40}	1	2	3	2	2	2
	3	3	3	1	2	3		3	3	3	3	2	3
	1	1	3	1	2	3		1	1	3	1	2	3
S_{41}	1	2	3	2	2	2	S_{42}	1	2	3	2	2	3
	3	3	3	3	3	3		3	3	3	3	2	3
	1	1	3	1	2	3		1	1	3	2	2	1
S_{43}	1	2	3	2	2	3	S_{44}	1	2	3	2	2	2
	3	3	3	3	3	3		3	3	3	1	2	3
	1	1	3	2	2	1		1	1	3	2	2	2
S_{45}	1	2	3	2	2	2	S_{46}	1	2	3	2	2	2
	3	3	3	2	2	3		3	3	3	1	2	3
	1	1	3	2	2	2		1	1	3	2	2	2
S_{47}	1	2	3	2	2	2	S_{48}	1	2	3	2	2	2
	3	3	3	2	2	1		3	3	3	2	2	2
	1	1	3	2	2	2		1	1	3	2	2	2
S_{49}	1	2	3	2	2	2	S_{50}	1	2	3	2	2	2
	3	3	3	2	2	3		3	3	3	3	3	3
	1	1	3	2	2	3		1	1	3	2	2	3
S_{51}	1	2	3	2	2	3	S_{52}	1	2	3	2	2	3
	3	3	3	2	2	3		3	3	3	3	3	3
	1	1	3	3	1	3		1	1	3	3	1	3
S_{53}	1	2	3	1	2	3	S_{54}	1	2	3	3	2	3
	3	3	3	3	3	3		3	3	3	3	3	3
	1	1	3	3	2	3		1	1	3	3	2	3
S_{55}	1	2	3	2	2	2	S_{56}	1	2	3	3	2	3
	3	3	3	3	2	3		3	3	3	3	2	3
	1	1	3	3	3	3		1	1	3	3	3	3
S_{57}	1	2	3	1	2	3	S_{58}	1	2	3	2	2	2
	3	3	3	3	3	3		3	3	3	3	3	3
	1	1	3	3	3	3		1	1	3	3	3	3
S_{59}	1	2	3	3	1	3	S_{60}	1	2	3	3	2	3
	3	3	3	3	3	3		3	3	3	3	3	3
	1	1	3	3	3	3							
S_{61}	1	2	3	3	3	3							
	3	3	3	3	3	3							

1. The proof of Theorem 1

In this section we shall provide the proof of Theorem 1. Let $\mathbf{HSP}(S)$ denote the variety generated by an ai-semiring S and $\underline{k}$ the set $\{1, 2, \ldots, k\}$ for a positive integer k. We start with a technique that will be used repeatedly in the sequel. Suppose that Σ is a set of identities which include the identities that determine **AI** and that $u \approx v$ is an **AI**-identity, where $u = u_1 + \cdots + u_k, v = v_1 + \cdots + v_\ell, u_i, v_j \in X^+, i \in \underline{k}, j \in \underline{\ell}$. Then it is easy to see that the ai-semring variety defined by $u \approx v$ is equal to the ai-semiring variety defined by the simpler identities $u \approx u + v_j, v \approx v + u_i, i \in \underline{k}, j \in \underline{\ell}$. Thus, to show that $u \approx v$ is derivable from Σ, we only need to show that $u \approx u + v_j, v \approx v + u_i, i \in \underline{k}, j \in \underline{\ell}$ can be derived from Σ.

Proposition 2. $\mathbf{HSP}(S_2)$ *is the ai-semiring variety determined by the identities*

$$x_1x_2x_3 \approx y_1y_2y_3, \tag{1.1}$$

$$x + x^2 \approx x^3, \tag{1.2}$$

$$x^2 + y^2 \approx xy, \tag{1.3}$$

$$x^3 + y \approx x^3. \tag{1.4}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to one of the following terms: $x_1 + \cdots + x_m$, $x_1^2 + \cdots + x_m^2$, $x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$ and x^3, where $x_1, \ldots, x_m$ are distinct variables, $y_1, \ldots, y_n$ are distinct variables, and $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$. Suppose that $u = u_1 + u_2 + \cdots + u_k$ is an **AI**-term, where $u_i \in X^+, i \in \underline{k}$. We shall show that there exists an **AI**-term u' in canonical form such that the identities (1.1)–(1.4) and the identities determining **AI** imply the identity $u \approx u'$. The following cases are needed.

- $\ell(u_i) = 1$ for all $i \in \underline{k}$. Then $u = x_1 + \cdots + x_m$.
- $\ell(u_i) = 2$ for all $i \in \underline{k}$. Then the identity (1.3) implies $u \approx x_1^2 + \cdots + x_m^2$.
- $\ell(u_i) \leq 2$ for all $i \in \underline{k}$, $\ell(u_{i_1}) = 1$ for some $i_1 \in \underline{k}$ and $\ell(u_{i_2}) = 2$ for some $i_2 \in \underline{k}$. If $c(u_i) \bigcap c(u_j) \neq \emptyset$ for some u_i and u_j with $\ell(u_i) = 1$ and $\ell(u_j) = 2$, then the identities (1.2)–(1.4) implies $u \approx x^3$. Otherwise, we have that the identity (1.3) implies $u \approx x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$.
- $\ell(u_i) \geq 3$ for some $i \in \underline{k}$. Then the identities (1.1) and (1.4) imply $u \approx x^3$.

It is routine to check that S_2 satisfies the identities (1.1)–(1.4). In the remainder we shall show that every identity which is satisfied in S_2 can be derived from the identities (1.1)–(1.4) and the identities determining **AI**. By the above arguments it is enough to show that if S_2 satisfies an identity $u \approx v$, where u and v are **AI**-terms in canonical forms, then the identities (1.1)–(1.4) and the identities determining **AI** imply $u \approx v$. Notice that T_2 can be embedded into S_2. We only need to consider the following cases:

- $u = x_1 + \cdots + x_m, v = y_1 + \cdots + y_n$. It is easy to see that $u \approx v$ is trivial.
- $u = x_1^2 + \cdots + x_m^2, v = y_1^2 + \cdots + y_n^2$. It is easy to see that $u \approx v$ is trivial.
- $u = x_1^2 + \cdots + x_m^2, v = y_1 + \cdots + y_k + z_1^2 + \cdots + z_\ell^2$. Let $\varphi : P_f(X^+) \to S_2$ be a semiring homomorphism such that $\varphi(x) = 3$ for every variable x in X. Then $\varphi(u) = 2$ and $\varphi(v) = 1$, a contradiction. Thus $u \approx v$ is not satisfied in S_2.
- $u = x_1^2 + \cdots + x_m^2, v = t^3$. This case is similar to the preceding one.

- $u = y_1 + \cdots + y_k + z_1^2 + \cdots + z_\ell^2, v = y'_1 + \cdots + y'_m + {z'}_1^2 + \cdots + {z'}_n^2$. It is easy to see that $u \approx v$ is trivial.
- $u = y_1 + \cdots + y_k + z_1^2 + \cdots + z_\ell^2, v = x^3$. Let $\varphi : P_f(X^+) \to S_2$ be a semiring homomorphism such that $\varphi(y_i) = 2$, $\varphi(z_j) = 3$ and $\varphi(t) = 1$ for all $i \in \underline{k}$, $j \in \underline{\ell}$. Then $\varphi(u) = 2$ and $\varphi(v) = 1$, a contradiction. Thus $u \approx v$ is not satisfied in S_2.
- $u = x_1^3, v = x_2^3$. Then the identity (1) implies $u \approx v$.

This completes the proof. □

Proposition 3. $\mathbf{HSP}(S_4)$ *is the ai-semiring variety determined by the identities*

$$xy \approx x^2 y, \tag{1.5}$$

$$xyz \approx yxz, \tag{1.6}$$

$$x + y^2 \approx xy^2, \tag{1.7}$$

$$x + yz \approx yx + yz. \tag{1.8}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to one of the following terms: $x_1 + \cdots + x_m$, $x_1^2 \cdots x_m^2$ and $x_1^2 \cdots x_m^2(y_1 + \cdots + y_n)$, where $x_1, \ldots, x_m$ are distinct variables, $y_1, \ldots, y_n$ are distinct variables and $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$. Suppose that $u = u_1 + u_2 + \cdots + u_k$ is an **AI**-term, where $u_i \in X^+$, $i \in \underline{k}$. We shall show that there exists an **AI**-term u' in canonical form such that the identities (1.5)–(1.8) and the identities determining **AI** imply the identity $u \approx u'$. The following cases are needed.

- $\ell(u_i) = 1$ for all $i \in \underline{k}$. Then $u = x_1 + \cdots + x_m$.
- $m(t(u_i), u_i) \geq 2$ for some $j \in \underline{k}$. Then the identities (1.5)–(1.7) imply $u \approx x_1^2 \cdots x_m^2$.
- $\ell(u_i) \geq 2$ for some $i \in \underline{k}$, $m(t(u_j), u_j) = 1$ for every $j \in \underline{k}$. Then the identities (1.5), (1.6) and (1.8) imply $u \approx x_1^2 \cdots x_m^2(y_1 + \cdots + y_n)$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$.

It is routine to check that S_4 satisfies the identities (1.5)–(1.8). By the above arguments it is enough to show that if S_4 satisfies an identity $u \approx v$, where u and v are **AI**-terms in canonical forms, then the identities (1.5)–(1.8) and the identities determining **AI** imply $u \approx v$. Since T_2 can be embedded into S_4, we only need to consider the following cases:

- $u = x_1 + \cdots + x_m, v = y_1 + \cdots + y_n$. It is easy to see that $u \approx v$ is trivial.
- $u = x_1^2 \cdots x_m^2, v = y_1^2 \cdots y_n^2$. It is easy to see that $u \approx v$ is trivial.
- $u = x_1^2 \cdots x_m^2, v = y_1^2 \cdots y_k^2(z_1 + \cdots + z_\ell)$. Let $\varphi : P_f(X^+) \to S_4$ be a semiring homomorphism such that $\varphi(y_i) = 3$, $\varphi(z_j) = 2$, $i \in \underline{k}$, $j \in \underline{\ell}$, $\varphi(x) = 1$ for every remaining variable x. Then $\varphi(u) = 1$ or 3, $\varphi(v) = 2$, a contradiction. Thus $u \approx v$ is not satisfied in S_4.
- $u = x_1^2 \cdots x_m^2(y_1 + \cdots + y_n)$, $v = z_1^2 \cdots z_k^2(t_1 + \cdots + t_\ell)$. It is easy to see that $u \approx v$ is trivial.

This completes the proof. □

Proposition 4. $\mathbf{HSP}(S_{44})$ *is the ai-semiring variety determined by the identities*

$$x^3 \approx x^2, \tag{1.9}$$

$$xy \approx yx, \tag{1.10}$$

$$x + xy \approx x, \tag{1.11}$$

$$x^2 y + xy^2 \approx xy. \tag{1.12}$$

P r o o f. An **AI**-term $u = u_1 + \cdots + u_n$ is said to be in canonical form if every u_i is equal to one of the following terms: $x, x_1^2 \cdots x_m^2$ and $x_1^2 \cdots x_m^2 y$, where $x_1, \ldots, x_m$ are distinct variables and $y \neq x_i$ for every $i \in \underline{m}$. Let $p = x_1 \cdots x_n$ be an element of X^+ such that $n \geq 2$. By induction on n we have that the identity (1.10) and (1.12) imply $p \approx \sum_{1 \leq i \leq n} x_1^2 \cdots x_{i-1}^2 x_{i+1}^2 \cdots x_n^2 x_i$. It follows that for any **AI**-term u, there exists an **AI**-term u' in canonical form such that (1.9)–(1.12) imply $u \approx u'$.

It is easy to check that S_{44} satisfies the identities (1.9)–(1.12). To show that $\mathbf{HSP}(S_{44})$ is determined by (1.9)–(1.12), by the above arguments it suffices to show that if S_{44} satisfies $u+p \approx u$, where $u+p$ and u are **AI**-terms in canonical forms, then the identities (1.9)–(1.12) and the identities determining **AI** imply $u + p \approx u$. We shall consider the following three cases.

- $p = x$. Since N_2 can be embedded into S_{44}, there exists some u_i in u such that $u_i = x$. It follows that $u + p \approx u$ is trivial.

- $p = x_1^2 \cdots x_m^2$. Since D_2 can be embedded into S_{44}, there exists some u_i in u such that $c(u_i) \subseteq c(p)$ and so (1.9) and (1.10) imply $u_i p \approx p$. Further, we have

$$u \approx u + u_i \overset{(1.11)}{\approx} u + u_i + u_i p \approx u + u_i + p \approx u + p.$$

- $p = x_1^2 \cdots x_m^2 y$. Since D_2 can be embedded into S_{44}, we have that $\{u_i \in u \mid c(u_i) \subseteq c(p)\}$ is non-empty. Suppose that for any u_i in $\{u_i \in u \mid c(u_i) \subseteq c(p)\}$, there exists x in $c(u_i)$ such that $m(x, p) < m(x, u_i)$. That is to say, $m(y, u_i) = 2$ for every u_i in $\{u_i \mid c(u_i) \subseteq c(p)\}$. Let $\varphi : P_f(X^+) \to S_{44}$ be a semiring homomorphism such that $\varphi(z) = 2$ for every $z \in X \setminus c(p)$, $\varphi(x_i) = 3$ for every $i \in \underline{m}$ and $\varphi(y) = 1$. Then $\varphi(u) = 2$ and $\varphi(u + p) = 1$, a contradiction. Thus there exists u_i in $\{u_i \in u \mid c(u_i) \subseteq c(p)\}$ such that $m(x, u_i) \leq m(x, p)$ for every x in $c(u_i)$. If $y \in c(u_i)$, then $m(y, u_i) = 1$ and so (1.9) and (1.10) imply $u_i x_1^2 \cdots x_m^2 \approx p$. Further, we have
$$u \approx u + u_i \overset{(1.11)}{\approx} u + u_i + u_i x_1^2 \cdots x_m^2 \approx u + p.$$
If $y \notin c(u_i)$, then (1.9) and (1.10) imply $u_i p \approx p$. We therefore have
$$u \approx u + u_i \overset{(1.11)}{\approx} u + u_i + u_i p \approx u + p.$$

This completes the proof. $\square$

Proposition 5. $\mathbf{HSP}(S_{46})$ *is the ai-semiring variety determined by the identities*

$$x^2 y \approx xy, \tag{1.13}$$
$$x^2 y^2 \approx y^2 x^2, \tag{1.14}$$
$$xyz \approx yxz, \tag{1.15}$$
$$x + xy \approx x, \tag{1.16}$$
$$x + yx \approx x. \tag{1.17}$$

P r o o f. An **AI**-term $u = u_1 + \cdots + u_n$ is said to be in canonical form if every u_i is equal to one of the following terms: $x, x_1^2 \cdots x_m^2$ and $x_1^2 \cdots x_m^2 y$, where $y \neq x_i$ for all $i \in \underline{m}$. Let p be an element of X^+ such that $\ell(p) \geq 2$. If $m(t(p), p) = 1$, then the identities (1.13)–(1.15) imply $p \approx x_1^2 \cdots x_m^2 y$. If $m(t(p), p) \geq 2$, then the identities (1.13)–(1.15) imply $p \approx x_1^2 \cdots x_m^2$. It follows that for any **AI**-term u, there exists an **AI**-term u' in canonical form such that (1.13)–(1.15) imply $u \approx u'$.

It is routine to check that S_{46} satisfies the identities (1.13)–(1.17). To show that $\mathbf{HSP}(S_{46})$ is the ai-semiring variety determined by (1.13)–(1.17), by the above arguments it suffices to show that if S_{46} satisfies $u+p \approx u$, where $u+p$ and u are **AI**-terms in canonical form, then (1.13)–(1.17) imply $u+p \approx u$. The following three cases are necessary.

- $p = x$. Since D_2 can be embedded into S_{46}, there exists some u_i in u such that $c(u_i) = \{x\}$. Suppose that $u_i = x^2$ for every u_i in u with $c(u_i) = \{x\}$. Let $\varphi : X \to S_{46}$ be a semiring homomorphism such that $\varphi(x) = 1$ and $\varphi(y) = 2$ for every $y \neq x$. Then $\varphi(u) = 2$ and $\varphi(u+p) = 1$, a contradiction. Thus there exists u_i in u such that $u_i = x$ and so $u+p \approx u$ is trivial.

- $p = x_1^2 \cdots x_m^2$. Since D_2 can be embedded into S_{46}, there exists some u_i in u such that $c(u_i) \subseteq c(p)$ and so (1.13)–(1.15) imply $p \approx u_i p$. We now have
$$u+p \approx u+u_i+p \approx u+u_i+u_ip \overset{(1.16)}{\approx} u+u_i \approx u.$$

- $p = x_1^2 \cdots x_m^2 y$. Since D_2 can be embedded into S_{46}, it follows that $\{u_i \in u \mid c(u_i) \subseteq c(p)\}$ is non-empty. Suppose that $m(y, u_i) = 2$ for every u_i in $\{u_i \in u \mid c(u_i) \subseteq c(p)\}$. Let $\varphi : P_f(X^+) \to S_{46}$ be a semiring homomorphism such that $\varphi(z) = 2$ for every $z \notin c(p)$, $\varphi(x_i) = 3$ for every $i \in \underline{m}$ and $\varphi(y) = 1$. Then $\varphi(u) = 2$ and $\varphi(u+p) = 1$, a contradiction. Thus we only need to consider the following cases:

  - ◇ $(\exists u_i \in \{u_i \in u \mid c(u_i) \subseteq c(p)\})\ y \notin c(u_i)$. Then
  $$u+p \approx u+u_i+p \overset{(1.13)\text{–}(1.15)}{\approx} u+u_i+u_ip \overset{(1.16)}{\approx} u+u_i \approx u.$$

  - ◇ $(\exists u_i \in \{u_i \in u \mid c(u_i) \subseteq c(p)\})\ y \in c(u_i)$, $t(u_i) = y$ and $m(y, u_i) = 1$. Then
  $$u+p \approx u+u_i+p \overset{(1.13)\text{–}(1.15)}{\approx} u+u_i+x_1^2 \cdots x_m^2 u_i \overset{(1.17)}{\approx} u+u_i \approx u.$$

This completes the proof. □

Proposition 6. $\mathbf{HSP}(S_{47})$ *is the ai-semiring variety determined by the identities*

$$xy \approx yx, \tag{1.18}$$
$$x + xy \approx x, \tag{1.19}$$
$$x^2 + xy \approx x^2, \tag{1.20}$$
$$x + x_1x_2x_3 \approx x. \tag{1.21}$$

P r o o f. It is easy to verify that S_{47} satisfies the identities (1.18)–(1.21). In the remainder it suffices to show that every identity which is satisfied in S_{47} is derivable from (1.18)–(1.21). Let $u+p \approx u$ be such an identity, where $u = u_1 + \cdots + u_m, u_i, p \in X^+, i \in \underline{m}$. We consider the following three cases.

- $\ell(p) = 1$. Since N_2 can be embedded into S_{47}, there exists u_i in u such that $u_i = p$. Thus $u+p \approx u$ is trivial.

- $\ell(p) = 2$. Suppose that for any u_i in u, $c(u_i) \nsubseteq c(p)$. Let $\varphi\colon P_f(X^+) \to S_{47}$ be a semiring homomorphism such that $\varphi(z) = 2$ for every $z \in X \setminus c(p)$ and $\varphi(x) = 3$ for every $x \in c(p)$. Then $\varphi(u) = 2$ and $\varphi(u + p) = 1$, a contradiction. Thus there exists u_i in u such that $c(u_i) \subseteq c(p)$. Assume that $\ell(u_i) \geq 3$ for every u_i in $\{u_i \in u \mid c(u_i) \subseteq c(p)\}$. Then $\varphi(u) = 2$ and $\varphi(u+p) = 1$, a contradiction. This implies that there exists u_i in u such that $c(u_i) \subseteq c(p)$ and $\ell(u_i) \leq 2$. Further, the identities (1.18)–(1.20) imply

$$u + p \approx u + u_i + p \approx u + u_i \approx u.$$

- $\ell(p) \geq 3$. Then $u + p \approx u$ can be derived from (1.21).

This completes the proof. □

Proposition 7. **HSP**(S_{53}) *is the ai-semiring variety determined by the identities*

$$xy \approx yx, \tag{1.22}$$
$$xy + y^2 \approx x + y^2, \tag{1.23}$$
$$x + xy \approx xy, \tag{1.24}$$
$$xy + yz + xz \approx xyz. \tag{1.25}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to one of the following terms: $x_1 + \cdots + x_m$, $x_1y_1 + \cdots + x_my_m$, and $x_1 + \cdots + x_m + y_1z_1 + \cdots + y_nz_n$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j, z_j \mid j \in \underline{n}\} = \emptyset$, x^2 and xy can not occur simultaneously. Suppose that $u = u_1 + u_2 + \cdots + u_k$ is an **AI**-term, where $u_i \in X^+, i \in \underline{k}$. It is easy to show that there exists an **AI**-term u' in canonical form such that the identities (1.22)–(1.25) imply the identity $u \approx u'$.

It is routine to check that S_{53} satisfies the identities (1.22)–(1.25). In the remainder we shall show that every identity which is satisfied in S_{53} can be derived from the identities (1.22)–(1.25). By the above arguments it suffices to show that if S_{53} satisfies an identity $u \approx v$, where u and v are **AI**-terms in canonical form, then the identities (1.22)–(1.25) and the identities determining **AI** imply $u \approx v$. Notice that T_2 can be embedded into S_{53}. We only need to consider the following cases:

- $u = x_1 + \cdots + x_m$, $v = y_1 + \cdots + y_n$. It follows immediately that $u \approx v$ is trivial.

- $u = x_1y_1 + \cdots + x_my_m$, $v = z_1s_1 + \cdots + z_ns_n$. For any $i \in \underline{m}$, suppose that $\{x_1, y_1\}$ is not equal to $\{z_j, s_j\}$ for every $j \in \underline{n}$. Consider the following two subcases:

  ◇ $x_i = y_i$. Let $\varphi : P_f(X^+) \to S_{53}$ be a semiring homomorphism such that $\varphi(x_i) = 1$ and $\varphi(z) = 2$ for every $z \in X \setminus \{x_i\}$. Then $\varphi(u) = 3$ and $\varphi(v) = 1$, a contradiction.

  ◇ $x_i \neq y_i$. Since M_2 can be embedded into S_{53}, we can deduce that S_{53} satisfies one of the following identities: $x_iy_i \approx x_i^2 + y_i^2$, $x_iy_i \approx x_i^2 + y_i$, $x_iy_i \approx x_i + y_i^2$ and $x_iy_i \approx x_i + y_i$, a contradiction.

  Thus $\{x_i, y_i\}$ is equal to $\{z_j, s_j\}$ for some $j \in \underline{n}$. Similarly, for any $j \in \underline{n}$, $\{z_j, s_j\}$ is equal to $\{x_i, y_i\}$ for some $i \in \underline{m}$. Hence $u \approx v$ is trivial.

- $u = x_1y_1 + \cdots + x_my_m, v = z_1 + \cdots + z_k + s_1t_1 + \cdots + s_\ell t_\ell$. We consider the following two subcases.

  ◇ $\{x_i, y_i\} \subseteq \{z_i \mid i \in \underline{k}\}$ for some $i \in \underline{m}$. Let $\psi : P_f(X^+) \to S_{53}$ be a semiring homomorphism such that $\psi(z_i) = 1$ for every $i \in \underline{k}$ and $\psi(x) = 2$ for every $z \in X \setminus \{z_i \mid i \in \underline{k}\}$. Then $\psi(u) = 3$ and $\psi(v) = 1$, a contradiction.

- ◇ $\{x_i, y_i\} \nsubseteq \{z_i \mid i \in \underline{k}\}$ for every $i \in \underline{m}$. Notice that M_2 can be embedded into S_{53}. Let $\theta : P_f(X^+) \to S_{53}$ be a semiring homomorphism such that $\theta(x_i) = \theta(y_i) = 1$ if $\{x_i, y_i\} \bigcap \{z_i \mid i \in \underline{k}\} \neq \emptyset$, and $\theta(y) = 2$ for every remaining variable y. Then $\theta(u) = 3$ and $\theta(v) = 1$, a contradiction.

This shows that $u \approx v$ is not satisfied in S_{53}.

- $u = x_1 + \cdots + x_m + y_1z_1 + \cdots + y_nz_n, v = x'_1 + \cdots + x'_k + y'_1z'_1 + \cdots + y'_\ell z'_\ell$. Suppose that $x_1 = y'_i$ for some i.
  - ◇ $y'_i = z'_i$. Choose every variable in $X \setminus \{x_1\}$ to 2. Then S_{53} satisfies $x_1 \approx x_1^2$, a contradiction.
  - ◇ $y'_i \neq z'_i$. Choose every variable in $X \setminus \{y'_i, z'_i\}$ to 2. Then S_{53} satisfies $y'_i + z'_i \approx y'_iz'_i$ or $y'_i + {z'_i}^2 \approx y'_iz'_i$, a contradiction.

  This implies that $x_1 + \cdots + x_m \approx x'_1 + \cdots + x'_k$ is trivial and so S_{53} satisfies $y_1z_1 + \cdots + y_nz_n \approx y'_1z'_1 + \cdots + y'_\ell z'_\ell$. By the preceding case it follows that $y_1z_1 + \cdots + y_nz_n \approx y'_1z'_1 + \cdots + y'_\ell z'_\ell$ is trivial. Hence $u \approx v$ is trivial.

This completes the proof. □

Proposition 8. **HSP**(S_{55}) *is the ai-semiring variety determined by the identities*

$$xy \approx yx, \tag{1.26}$$
$$xy \approx x^2y, \tag{1.27}$$
$$xy \approx xy + xyz, \tag{1.28}$$
$$x^2 \approx x^2 + x. \tag{1.29}$$

P r o o f. It is routine to check that S_{55} satisfies (1.26)–(1.29). In the remainder it suffices to show that if S_{55} satisfies $u \approx u+q$, where $u = u_1+u_2+\cdots+u_m, u_i, q \in X^+, i \in \underline{m}$, then (1.26)–(1.29) and the identities determining **AI** imply the identity $u \approx u+q$. Choose $Z = (\bigcup_{i\in \underline{m}} c(u_i))\setminus c(q)$. By [21, Lemma 2.11] we have that T_2 satisfies $D_Z(u) \approx D_Z(u) + q$, where $D_Z(u)$ denotes the sum of terms u_i for which $c(u_i) \subseteq c(q)$. We may assume that $D_Z(u) = u_1 + u_2 + \cdots + u_k$. The following two cases are necessary.

- $\ell(q) = 1$. Then (1.27) and (1.29) implies $u \approx u + q$.
- $\ell(q) \geq 2$. Then there exists u_i with $c(u_i) \subseteq c(q)$ such that $\ell(u_i) \geq 2$. Further, by (1.26)–(1.28) we have
$$u \approx u + u_i \overset{(1.28)}{\approx} u + u_i + u_iq \overset{(1.26),(1.27)}{\approx} u + u_i + q \approx u + q.$$

This completes the proof. □

Proposition 9. **HSP**(S_{57}) *is the ai-semiring variety determined by the identities*

$$xyz \approx yxz, \tag{1.30}$$
$$x^2y \approx xy, \tag{1.31}$$
$$x + yz \approx yx + yz, \tag{1.32}$$
$$x^2 + xy \approx xy. \tag{1.33}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to $x_1 + \cdots + x_m$, x^2 or $x_1 \cdots x_m(y_1 + \cdots + y_n)$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$. Let u be an arbitrary **AI**-term. It is easy to see that there exists an **AI**-term u' in canonical form such that (1.30)–(1.33) imply $u \approx u'$.

It is routine to check that S_{57} satisfies (1.30)–(1.33). In the remainder it is enough to show that if S_{57} satisfies $u \approx v$ where u and v are **AI**-terms in canonical form, then (1.30)–(1.33) and the identities determining **AI** imply $u \approx v$. Notice that both M_2 and T_2 can be embedded into S_{57}. We consider the following nontrivial case that $u = x_1 \cdots x_m(y_1 + \cdots + y_n)$, $v = z_1 \cdots z_k(t_1 + \cdots + t_\ell)$. For a fixed x_i, suppose that it is not equal to z_j for every $j \in \underline{k}$. Since M_2 can be embedded into S_{57}, it follows that x_i must be equal to some t_j. Choose x_i to 1 and every other variable to 2. We have that $3 = 1$, a contradiction. Thus $x_1 \cdots x_m \approx z_1 \cdots z_k$ is trivial. Choose x_i to 2 for every $i \in \underline{m}$. Then S_{57} satisfies $y_1 + \cdots + y_n \approx t_1 + \cdots + t_\ell$. Thus $y_1 + \cdots + y_n \approx t_1 + \cdots + t_\ell$ is trivial and so is $u \approx v$. □

Proposition 10. **HSP**(S_{58}) *is the ai-semiring variety determined by the identities*

$$xy \approx x^2, \tag{1.34}$$
$$x^2 \approx x + x^2. \tag{1.35}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to $x_1 + \cdots + x_m$, $y_1^2 + \cdots + y_n^2$ or $x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$. Let u be an arbitrary **AI**-term. It is easy to see that there exists an **AI**-term u' in canonical form such that (1.34) and (1.35) imply $u \approx u'$.

It is routine to check that S_{58} satisfies (1.34) and (1.35). In the remainder it is enough to show that if S_{58} satisfies $u \approx v$, where u and v are **AI**-terms in canonical form, then (1.34), (1.35) and the identities determining **AI** imply $u \approx v$. Notice that T_2 can be embedded into S_{58}. The following two cases are necessary.

- $u = x_1 + \cdots + x_m$, $v = y_1 + \cdots + y_n$. Then $u \approx v$ is trivial.
- $u = x_1^2 + \cdots + x_m^2$, $v = y_1^2 + \cdots + y_n^2$. Since L_2 can be embedded into S_{58}, it follows that $u \approx v$ is trivial.
- $u = x_1^2 + \cdots + x_k^2$, $v = y_1 + \cdots y_m + z_1^2 + \cdots + z_n^2$. Let $\psi : P_f(X^+) \to S_{58}$ be a semiring homomorphism such that $\psi(y_i) = 1$ for every $i \in \underline{m}$ and $\psi(x) = 2$ for every remaining variable x. Since L_2 can be embedded into S_{58}, it follows that $\psi(u) = 3$ and $\psi(v) = 1$, a contradiction.
- $u = x_1 + \cdots x_m + y_1^2 + \cdots + y_n^2$, $v = z_1 + \cdots z_k + t_1^2 + \cdots + t_\ell^2$. Since L_2 can be embedded into S_{58}, we have

$$\{x_i \mid i \in \underline{m}\} \bigcup \{y_i \mid i \in \underline{n}\} = \{z_j \mid j \in \underline{k}\} \bigcup \{t_j \mid j \in \underline{\ell}\}.$$

  For a fixed x_i, suppose that it is not equal to z_j for every $j \in \underline{k}$. Then x_i must be equal to some t_j. Choose x_i to 1 and every remaining variable to 2. We have that $1 = 3$, a contradiction. Thus x_i is equal to some z_i and so $\{x_i \mid i \in \underline{m}\} = \{z_j \mid j \in \underline{k}\}$. Hence $u \approx v$ is trivial.

This completes the proof. □

Proposition 11. $\mathbf{HSP}(S_{59})$ *is the ai-semiring variety determined by the identities*

$$x_1x_2x_3 \approx y_1y_2y_3, \tag{1.36}$$
$$x^3 + y \approx x^3, \tag{1.37}$$
$$x^2 + y^2 \approx xy, \tag{1.38}$$
$$x + x^2 \approx x^2. \tag{1.39}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to x^3, $x_1 + \cdots + x_m$, $y_1^2 + \cdots + y_n^2$ or $x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$. Let u be an arbitrary **AI**-term. It is easy to see that there exists an **AI**-term u' in canonical form such that (1.36)-(1.39) imply $u \approx u'$.

It is routine to check that S_{59} satisfies (1.36)-(1.39). In the remainder it is enough to show that if S_{59} satisfies $u \approx v$, where u and v are **AI**-terms in canonical form, then (1.36)-(1.39) and the identities determining **AI** imply $u \approx v$. Notice that T_2 can be embedded into S_{59}. The following cases are necessary.

- $u = x_1 + \cdots + x_m$, $v = y_1 + \cdots + y_n$. It is easy to see that $u \approx v$ is trivial.
- $u = t_1^3$, $v = t_2^3$. Then (1.36) implies $u \approx v$.
- $u = t_1^3$, $v = y_1^2 + \cdots + y_n^2$. Choose every variable to 2. Then $3 = 1$, a contradiction. Thus S_{59} does not satisfy $u \approx v$.
- $u = t_1^3$, $v = x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$. Choose every variable to 2. Then $3 = 1$, a contradiction. Thus S_{59} does not satisfy $u \approx v$.
- $u = x_1^2 + \cdots + x_m^2$, $v = y_1^2 + \cdots + y_n^2$. It is easy to see that $u \approx v$ is trivial.
- $u = x_1^2 + \cdots + x_m^2$, $v = y_1 + \cdots + y_k + z_1^2 + \cdots + z_\ell^2$. Consider the following two subcases.
  - ◇ $\{y_i \mid i \in \underline{k}\} \nsubseteq \{x_i \mid i \in \underline{m}\}$. Choose y_i to 3, where $y_i \notin \{x_i \mid i \in \underline{m}\}$. Choose every other variable to 2. Then 1=3, a contradiction.
  - ◇ $\{y_i \mid i \in \underline{k}\} \subseteq \{x_i \mid i \in \underline{m}\}$. Choose y_i to 1 for every $i \in \underline{m}$ and every other variable to 2. Then 3=1, a contradiction.

  Thus S_{59} does not satisfy $u \approx v$.
- $u = x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$, $v = z_1 + \cdots + z_k + s_1^2 + \cdots + s_\ell^2$. Fix x_i. Suppose that x_i is not equal to z_j for every $j \in \underline{k}$.
  - ◇ $x_i \in \{s_j \mid j \in \underline{\ell}\}$. Choose x_i to 1 and every other variable to 2. Then 1=3, a contradiction.
  - ◇ $x_i \notin \{s_j \mid j \in \underline{\ell}\}$. Choose x_i to 3 and every other variable to 2. Then 3=1, a contradiction.

  Thus x_i is equal to z_j for some $j \in \underline{k}$ and so $x_1 + \cdots + x_m \approx z_1 + \cdots + z_k$ is trivial. Further, $y_1^2 + \cdots + y_n^2 \approx s_1^2 + \cdots + s_\ell^2$ holds in S_{59}. By the case 5 we have that this identity is trivial and so is $u \approx v$.

This complete the proof. □

Proposition 12. **HSP**(S_{60}) *is the ai-semiring variety determined by the identities*

$$x^3 \approx x^2, \tag{1.40}$$
$$x^2 + y^2 \approx xy, \tag{1.41}$$
$$x + x^2 \approx x^2. \tag{1.42}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to $x_1 + \cdots + x_m$, $y_1^2 + \cdots + y_n^2$ or $x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$. Let u be an arbitrary **AI**-term. It is easy to see that there exists an **AI**-term u' in canonical form such that (1.40)-(1.42) imply $u \approx u'$.

It is routine to check that S_{60} satisfies (1.40)-(1.42). In the remainder it suffices to show that if S_{60} satisfies $u \approx v$, where u and v are terms in canonical form, then (1.40)-(1.42) and the identities determining **AI** imply $u \approx v$. Notice that T_2 can be embedded into S_{60}. The following cases are necessary.

- $u = x_1 + \cdots + x_m$, $v = y_1 + \cdots + y_n$. It is easy to see that $u \approx v$ is trivial.
- $u = x_1^2 + \cdots + x_m^2$, $v = y_1^2 + \cdots + y_n^2$. Since M_2 can be embedded into S_{60}, it follows that $u \approx v$ is trivial.
- $u = x_1^2 + \cdots + x_m^2, v = y_1 + \cdots + y_k + z_1^2 + \cdots + z_\ell^2$. Then $\{y_i \mid i \in \underline{k}\} \subseteq \{x_i \mid i \in \underline{m}\}$. Choose y_i to 1 for every $i \in \underline{k}$ and every other variable to 2. Then 3=1, a contradiction. Thus S_{60} does not satisfy $u \approx v$.
- $u = x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$, $v = z_1 + \cdots + z_k + s_1^2 + \cdots + s_\ell^2$. Then

  $$\{x_i \mid i \in \underline{m}\} \bigcup \{y_i \mid i \in \underline{n}\} = \{z_j \mid j \in \underline{k}\} \bigcup \{s_j \mid j \in \underline{\ell}\}.$$

  Fix x_i. Suppose that x_i is not equal to z_j for all $j \in \underline{k}$. Choose x_i to 1 for all $i \in \underline{k}$ and every other variable to 2. Then 1=3, a contradiction. Thus $x_i = z_j$ for some $j \in \underline{k}$. This implies that $u \approx v$ is trivial.

This complete the proof. □

By Propositions 1–12 we immediately complete the proof of Theorem 1.

2. Conclusion

We have answered the finite basis problem for all ai-semirings of order three except S_7. This will lay a solid foundation for our subsequent work about ai-semiring varieties. Moreover, we conjecture that the semiring S_7 is nonfinitely based. In contrast to the rich results in the theory of semigroup varieties [26], there are still many problems to be solved in the theory of semiring varieties. In particular, it is of the interest to study the variety generated by all ai-semirings of order three.

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